

PRIMITIVNI' FUNKCE (necessity/ integral)

I. Defi'ice: Funkce F je primitivni' funkce k f na intervalu (a,b) , kdyz' plat'

$$\underline{F'(x) = f(x), x \in (a,b)}$$

II. Existence!

1) f je spojita' funkce na $(a,b) \Rightarrow$ na (a,b) existuji k f primitivni' fce

2) F, G jsou primitivni' k f na $(a,b) \Rightarrow \Rightarrow \exists c \in \mathbb{R}$ tak, ze'

$$G(x) = F(x) + c, x \in (a,b),$$

tedy, k f existuji nekonecne' mnoho primitivni'ch funkc' (pokud \exists f na (a,b) primitivni' fce) a kazde' dve' se li'si o konstantu;

zovodne
$$F(x) + c = \int f(x) dx$$

(necessity/ integral)

III. Vysvet'

A) Tabulka integralu:

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \alpha \neq -1 \quad x \in \mathbb{R} \quad (\frac{1}{2} \subset \mathcal{D}(x^\alpha))$$

$$\int \frac{1}{x} dx = \ln|x| + c, x \in (-\infty, 0), x \in (0, +\infty)$$

$$\int e^x dx = e^x + c, x \in \mathbb{R}$$

$$\int \sin x dx = -\cos x + c, x \in \mathbb{R}$$

$$\int \cos x dx = \sin x + c, x \in \mathbb{R}$$

$$\int \frac{1}{1+x^2} dx = \arctan x + c, x \in \mathbb{R}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c, x \in (-1,1)$$

B) Je-li $\int f(x) dx = F(x) + C$ na intervalu J , pak

$$(a \neq 0) \quad \int f(ax+b) dx = \frac{1}{a} F(ax+b) + C \quad (\text{ne odměňuje intervalu})$$

Pr. $\int e^{3x+1} dx = \frac{1}{3} e^{3x+1} + C, x \in \mathbb{R}$

$$\int \frac{1}{2-5x} dx = -\frac{1}{5} \ln|2-5x| + C, x \neq \frac{2}{5}$$

$$\int \frac{1}{1+4x^2} dx = \int \frac{1}{1+(2x)^2} dx = \frac{1}{2} \arctan(2x) + C, x \in \mathbb{R}$$

c) Vlastnosti nezáporného integrálu, které lze využít při výpočtu
(f, g spojitě $r(a, b)$)

1) $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx, x \in (a, b)$

2) $\int c f(x) dx = c \int f(x) dx, x \in (a, b)$

Pr. $\int \frac{x-1}{x^2} dx = \int \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \int \frac{1}{x} dx - \int \frac{1}{x^2} dx = \ln|x| + \frac{1}{x} + C$
 $x \in (-\infty, 0), x \in (0, +\infty)$

$$\int \sin^2 x dx = \int \frac{1 - \cos 2x}{2} dx = \frac{1}{2} \left(\int 1 dx - \int \cos 2x dx \right) =$$

$$= \frac{1}{2} \left(x - \frac{\sin 2x}{2} \right) + C = \frac{1}{2} \left(x - \sin x \cos x \right) + C, x \in \mathbb{R}$$

D) integrace per partes (f', g' spojitě $r(a, b)$)

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx \quad r(a, b)$$

Pr. 1) $\int x \sin x dx = \left| \begin{array}{l} f' = \sin x, f = -\cos x \\ g = x, g' = 1 \end{array} \right| = -x \cos x + \int \cos x dx =$
 $= -x \cos x + \sin x + C$

2) $\int \ln x dx = \int 1 \ln x dx = \left| \begin{array}{l} f' = 1, f = x \\ g = \ln x, g' = \frac{1}{x} \end{array} \right| = x \ln x - \int x \cdot \frac{1}{x} dx =$
 $x \ln x - x + C$

$$3, \int \sin^2 x dx = \int \sin x \cdot \sin x dx = \left| \begin{array}{l} f' = \sin x, f = -\cos x \\ g = \sin x, g' = \cos x \end{array} \right| =$$

$$= -\sin x \cdot \cos x + \int \cos^2 x dx = -\sin x \cdot \cos x + \int (1 - \sin^2 x) dx \Rightarrow$$

$$\Rightarrow 2 \int \sin^2 x dx = -\sin x \cos x + x, \text{ def}$$

$$\underline{\int \sin^2 x dx = \frac{1}{2} (x - \sin x \cos x) + C, x \in \mathbb{R}}$$

E) integrare prin metoda substituției

I. f este primitivă pe (a, b) , g' este primitivă pe (α, β) , $g(\alpha, \beta) = (a, b)$:

x -li $\int f(t) dt = F(t) + C$ pe (a, b) , x

$$\underline{\int f(g(x)) \cdot g'(x) dx = F(g(x)) + C, x \in (\alpha, \beta)}$$

Pr. 1) $\int_{x \in \mathbb{R}} e^{-x^2} (-2x) dx = \left| \begin{array}{l} -x^2 = t \\ -2x dx = dt \end{array} \right| = \int e^t dt = e^t + C = \underline{e^{-x^2} + C}$

2) $\int \frac{\ln(1+\sqrt{x})}{\sqrt{x}} dx = 2 \int \frac{\ln(1+\sqrt{x})}{2\sqrt{x}} dx = \left| \begin{array}{l} 1+\sqrt{x} = t \\ \frac{1}{2\sqrt{x}} dx = dt \end{array} \right| = \int \ln t dt =$
 $= t \ln t - t + C = \underline{(1+\sqrt{x})(\ln(1+\sqrt{x}) - 1) + C, x \in (0, +\infty)}$

II. f este primitivă pe (a, b) , g' este primitivă pe (α, β) , $g' \neq 0$ pe (α, β) , $g(\alpha, \beta) = (a, b)$.

pe (α, β) , x -li $\int f(g(t)) \cdot g'(t) dt = G(t) + C, t \in (\alpha, \beta)$

$$\underline{\int f(x) dx = G(g^{-1}(x)) + C, x \in (a, b)}$$

Pr. 1) $\int_{x \in (-1, 1)} \sqrt{1-x^2} dx = \left| \begin{array}{l} x = \sin t, t \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ dx = \cos t dt (\cos t > 0) \\ t = \arcsin x \end{array} \right| = \int \sqrt{1-\sin^2 t} \cdot \cos t dt =$

$$= \int \cos^2 t dt = \int \frac{1 + \cos 2t}{2} dt = \frac{1}{2} (t + \sin 2t) =$$

$$= \underline{\frac{1}{2} (\arcsin x + x \cdot \sqrt{1-x^2}) + C}$$

$$2) \int \frac{\sqrt{x}}{x+1} dx = \left| \begin{array}{l} \sqrt{x} = t \\ x = t^2 \\ dx = 2t dt \end{array} \right| = \int \frac{t \cdot 2t}{t^2+1} dt = 2 \int \frac{t^2}{t^2+1} dt =$$

$$= 2 \int \left(1 - \frac{1}{1+t^2}\right) dt = \underline{2(t - \arctan t) + C = 2(\sqrt{x} - \arctan \sqrt{x}) + C}$$

jestli nelze puvlodek & I:

$$3) \int \frac{k}{\sqrt{1+x^2}} dx = \left| \begin{array}{l} 1+x^2 = t \\ 2x dx = dt \end{array} \right| = \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + C = \underline{\sqrt{1+x^2} + C, x \in \mathbb{R}}$$

$$4) \int \frac{g'(x)}{g(x)} dx = \left| \begin{array}{l} g(x) = t \\ g'(x) dx = dt \end{array} \right| = \int \frac{1}{t} dt = \ln|t| + C = \ln|g(x)| + C,$$

$x \in (a,b), g, g' \text{ spojite' v } (a,b), g(x) \neq 0 \forall x \in (a,b)$

$$5) \int \frac{2x}{1+x^2} dx = \left| \begin{array}{l} 1+x^2 = t \\ 2x dx = dt \end{array} \right| = \int \frac{1}{t} dt = \ln|t| + C = \ln(1+x^2) + C,$$

$x \in \mathbb{R}$

(nelze puvlodek podle vzorce v puvl. 4)

F) Integrace racionálních funkcí

1) jednoduché (parciální) zlomky

$$(i) \int \frac{1}{x-a} dx = \ln|x-a| + C, \quad x \neq a$$

$$(ii) \int \frac{1}{(x-a)^n} = \frac{1}{1-n} \cdot \frac{1}{(x-a)^{n-1}} + C, \quad x \neq a$$

$n > 1, n \in \mathbb{N}$

$$(iii) \int \frac{Ax+B}{x^2+px+q} dx = \frac{A}{2} \int \frac{2x+p}{x^2+px+q} dx + \left(B - \frac{Ap}{2}\right) \int \frac{1}{\left(x+\frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)} dx =$$

$p^2 - 4q < 0$

$$= \frac{A}{2} \ln(x^2+px+q) + C \int \frac{1}{y^2+1} dy$$

substituce \downarrow podle vzorce
(viz puvlodek dale)

$$\begin{aligned}
 \text{Pr. } \int \frac{x-1}{x^2+4x+8} dx &= \frac{1}{2} \int \frac{2x+4}{x^2+4x+8} dx + (-3) \int \frac{1}{(x+2)^2+4} dx \\
 &= \frac{1}{2} \ln(x^2+4x+8) - \frac{3}{4} \int \frac{1}{\left(\frac{x+2}{2}\right)^2+1} dx = \\
 &= \frac{1}{2} \ln(x^2+4x+8) - \frac{3}{4} \cdot 2 \arctg\left(\frac{x+2}{2}\right) + C, x \in \mathbb{R}
 \end{aligned}$$

2) "návod" pro integraci racionálních funkcí

$$\int \frac{f(x)}{q(x)} dx, \quad f(x), q(x) - \text{mnohočleny, } \text{st } q(x) \geq 1$$

(i) st $f(x) \geq \text{st } q(x)$, pak

$$\frac{f(x)}{q(x)} = \tilde{f}(x) + \frac{r(x)}{q(x)},$$

kde $\text{st } \tilde{f} = \text{st } f - \text{st } q$ a
 $\text{st } r < \text{st } q$

(ii) st $f(x) < \text{st } q(x)$

$q(x)$ se rozdělí na součin
 lineárních činitelů, které
 odpovídají reálným kořenům
 a kvadratickým trojčlenům,
 které nejsou reálné kořeny

(iii) $\frac{f(x)}{q(x)}$ se rozdělí na součet
 racionálních zlomků tvaru:

$$(x-d)^m \rightarrow \frac{A_1}{x-d} + \frac{A_2}{(x-d)^2} + \dots + \frac{A_m}{(x-d)^m}$$

$m=1, 2, \dots$

$$(x^2+px+q)^n \rightarrow \frac{B_1x+C_1}{x^2+px+q} + \dots + \frac{B_nx+C_n}{(x^2+px+q)^n}$$

$$\text{Pr. } \frac{x^4-3x^3+5x^2-2x+1}{x^3-3x^2+4x-2} = x + \frac{x^2+1}{x^2-3x^2+4x-2}$$

(deletem)

$$q(x) = x^3-3x^2+4x-2 \text{ má kořen } d=1,$$

tedy

$$q(x) = (x-1)(x^2-2x+2),$$

polynom x^2-2x+2 má pouze reálné
 kořeny

$$\text{zde: } \frac{x^2+1}{x^3-3x^2+4x-2} = \frac{x^2+1}{(x-1)(x^2-2x+2)}$$

$$\frac{A}{x-1} + \frac{Bx+C}{x^2-2x+2}$$

(iv) rovnost' koeficientu
 vzhledem se uží metodou
 nerušených koeficientů

$$x^2+1 = A(x^2-2x+2) + (Bx+C)(x-1)$$

$$x^2+1 = (A+B)x^2 + (-2A+C-B)x + 2A-C$$

$$\text{př. } x^2: A+B = 1$$

$$x: -2A+B+C = 0$$

$$x^0: 2A - C = 1$$

$$\text{odhad: } \underline{A=2, B=-1, C=3}$$

(v) dále integrace jednotlivých
 racionálních zlomků

Tedy: $\int \frac{x^4 - 3x^3 + 5x^2 - 2x + 1}{x^3 - 3x^2 + 4x - 2} dx = \int x dx + \int \frac{x^2+1}{x^3-3x^2+4x-2} dx =$

(ii), (iii) $\frac{x^2}{2} + \int \frac{2}{x-1} dx + \int \frac{-x+3}{x^2-2x+2} dx =$

$$= \frac{x^2}{2} + 2 \ln|x-1| + \left(-\frac{1}{2}\right) \int \frac{2x-2}{x^2-2x+2} dx + 2 \int \frac{1}{(x-1)^2+1} dx$$

$$= \underline{\underline{\frac{x^2}{2} + 2 \ln|x-1| - \frac{1}{2} \ln(x^2-2x+2) + 2 \arctan(x-1) + C}}$$

$$x \in (-\infty, 1) \text{ nebo } x \in (1, +\infty)$$

6) Substituce, vedoucí ke integraci racionálních funkcí
 ($R(t)$ jsou racionální funkce)

$$1) \int R(e^x) dx = \left| \begin{array}{l} e^x = t \\ x = \ln t \\ dx = \frac{1}{t} dt \end{array} \right| = \int R(t) \cdot \frac{1}{t} dt$$

Pr. $\int \frac{e^x - 1}{e^{2x} - 2e^x + 2} dx = \int \frac{t-1}{t^2-2t+2} \cdot \frac{1}{t} dt = \dots$

$$2) \int R(\sin x, \frac{1}{x}) dx = \left| \begin{array}{l} \sin x = t \\ \frac{1}{x} dx = dt \end{array} \right| = \int R(t) dt = \dots$$

Pr.

$$\int \frac{\sin x}{\sin^2 x + 1} \cdot \frac{1}{x} dx = \int \frac{t}{t^2 + 1} dt = \dots$$

$$3) \int R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) dx = \left| \begin{array}{l} \sqrt[n]{\frac{ax+b}{cx+d}} = t \\ \dots \end{array} \right| = \dots$$

Pr.

$$\int \frac{3\sqrt{x}+1}{x(x+2\sqrt{x}+2)} dx = \left| \begin{array}{l} \sqrt{x} = t \\ x = t^2 \\ dx = 2t dt \end{array} \right| = \int \frac{3t+1}{t^2(t^2+2t+2)} \cdot 2t dt = \dots$$

$$4) \int_{a>0} R(x, \sqrt{ax^2+bx+c}) dx = \left| \begin{array}{l} \sqrt{ax^2+bx+c} = \sqrt{a}x \pm t \\ \text{(Euler's substitution)} \\ x = \dots \\ dx = \dots \end{array} \right|$$

$$5) a) \int R(\sin x, \cos x) dx = \left| \begin{array}{l} \text{if } \frac{t}{2} = t, \quad \sin x = \frac{2t}{1+t^2} \\ dx = \frac{2}{1+t^2} dt, \quad \cos x = \frac{1-t^2}{1+t^2} \\ x \in ((2k-1)\pi, (2k+1)\pi) \\ \text{other intervals of } R(t) \end{array} \right|$$

Pr.

$$\int \frac{1}{2+\cos x} dx = \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt = 2 \int \frac{1}{3+t^2} dt = \dots$$

(Dom. : $t \in (-\pi, \pi)$,

date for periodic a v lodech $x = (2k+1)\pi$ z' hiba
 "punctime" functiei "sleepit")

b) Je-li $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, (je R je lidek' a sine),
sae substitucí $\cos x = t$:

Pr. $\int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx = \int \frac{\sin x dx}{1 - \cos^2 x} = - \int \frac{dt}{1 - t^2} = \dots$

Podobně, pro $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ (je lidek' a cosine),
sae substitucí $\sin x = t$:

c) Je-li $R(-\sin x, -\cos x) = R(\sin x, \cos x)$,
sae substitucí $\lg x = t$

(i pro $\int R(\lg x) dx$ - substituce $\lg x = t$)

Pr. 1) $\int \frac{1}{\sin^2 x \cos^2 x} dx = \left| \begin{array}{l} \lg x = t \\ \frac{1}{\cos^2 x} dx = dt \end{array} \right| = \left(\begin{array}{l} \sin^2 x = \frac{t^2}{1+t^2} \\ \cos^2 x = \frac{1}{1+t^2} \end{array} \right)$
 $x \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$
 $= \int \frac{1+t^2}{t^2} dt = \dots$

2) $\int \frac{1}{1+\lg x} dx = \left| \begin{array}{l} \lg x = t \\ dx = \frac{1}{1+t^2} dt \end{array} \right| =$
 $x \in (\dots)$
 $= \int \frac{1}{1+t} \cdot \frac{1}{1+t^2} dt = \dots$